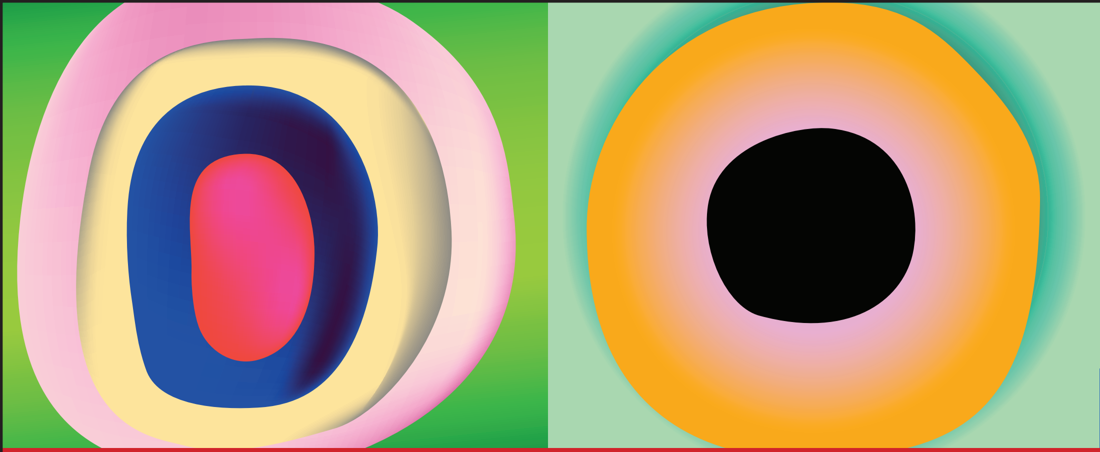
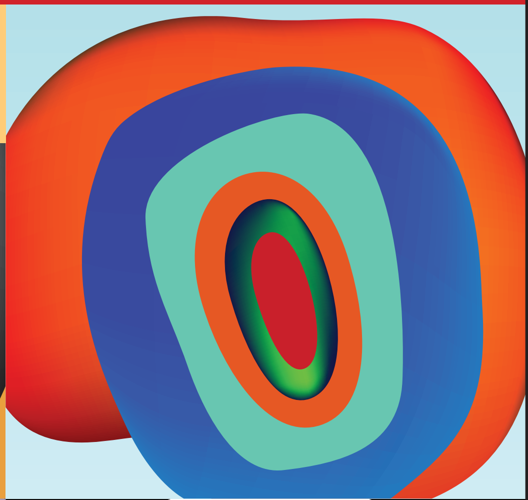



ninth edition

excursions in modern mathematics

peter tannenbaum



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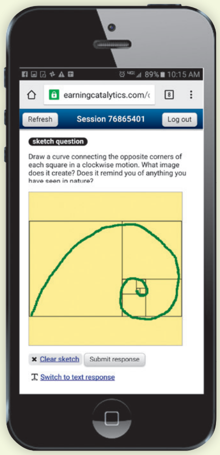

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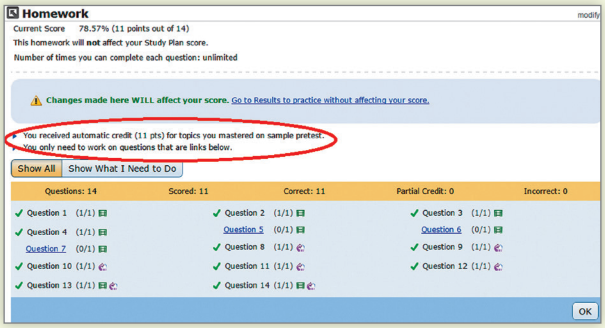

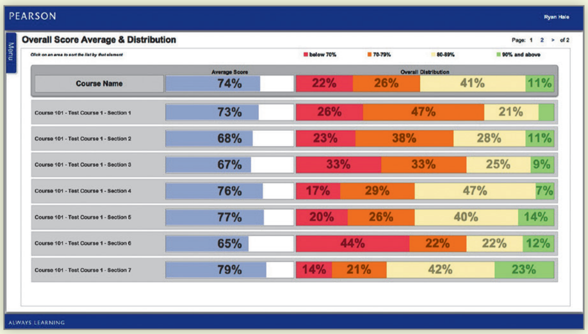

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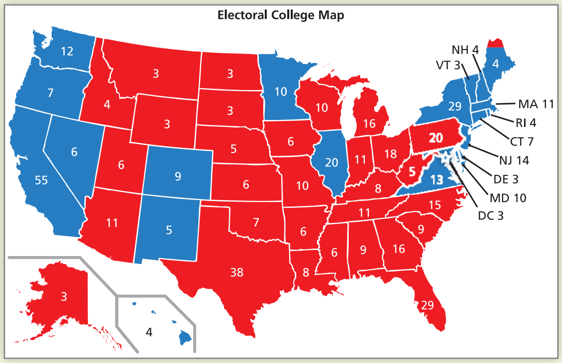

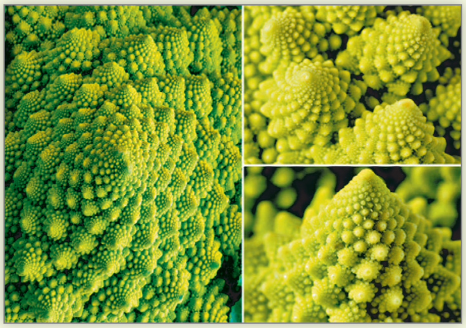

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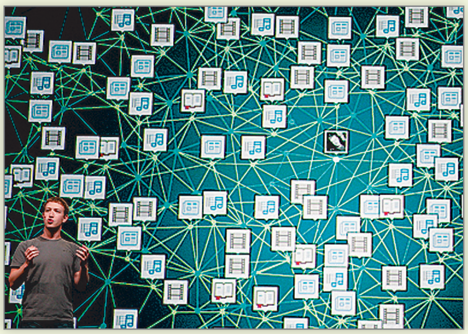

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Excursions in Modern Mathematics

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9th edition

Excursions in Modern Mathematics

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To the members of the board of Last Tango

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Contents

Preface

> "To most outsiders, modern mathematics is unknown territory. Its borders are protected by dense thickets of technical terms; its landscapes are a mass of indecipherable equations and incomprehensible concepts. Few realize that the world of modern mathematics is rich with vivid images and provocative ideas."
>
> – *Ivars Peterson,*
> The Mathematical Tourist

FROM THE AUTHOR

This text started more than 20 years ago as a set of lecture notes for a new, experimental "math appreciation" course (these types of courses are described, sometimes a bit derisively, as "math for poets"). Over time, the lecture notes grew into a text and the "poets" turned out to be social scientists, political scientists, economists, psychologists, environmentalists, and many other "ists." Over time, and with the input of many users, the contents have been expanded and improved, but the underlying philosophy of the text has remained the same since those handwritten lecture notes were handed out to my first group of students.

Excursions in Modern Mathematics is a travelogue into that vast and alien frontier that many people perceive mathematics to be. My goal is to show the open-minded reader that mathematics is a lively, interesting, useful, and surprisingly rich human activity.

The "excursions" in *Excursions* represent a collection of topics chosen to meet the following simple criteria.

- **Applicability.** There is no need to worry here about that great existential question of college mathematics: What is this stuff good for? The connection between the mathematics presented in these excursions and down-to-earth, concrete real-life problems is transparent and immediate.
- **Accessibility.** As a general rule, the excursions in this text do not presume a background beyond standard high school mathematics—by and large, intermediate algebra and a little Euclidean geometry are appropriate and sufficient prerequisites. (In the few instances in which more advanced concepts are unavoidable, an effort has been made to provide enough background to make the material self-contained.) A word of caution—this does not mean that the excursions in this book are easy! In mathematics, as in many other walks of life, simple and basic are not synonymous with easy and superficial.
- **Modernity.** Unlike much of traditional mathematics, which is often hundreds of years old, most of the mathematics in this text has been discovered within the last 100 years, and in some cases within the last couple of decades. Modern mathematical discoveries do not have to be the exclusive province of professional mathematicians.
- **Aesthetics.** The notion that there is such a thing as beauty in mathematics is surprising to most casual observers. There is an important aesthetic component in mathematics, and just as in art and music (which mathematics very much resembles), it often surfaces in the simplest ideas. A fundamental objective of this text is to develop an appreciation of the aesthetic elements of mathematics.

Outline

The excursions are organized into five independent parts, each touching on a different area where mathematics and the real world interface.

- **PART 1 Social Choice.** This part deals with mathematical applications to politics, social science, and government. How are *elections* decided? (Chapter 1);

How can the *power* of individuals, groups, or voting blocs be measured? (Chapter 2); How can assets commonly owned be *divided* in a *fair* and equitable manner? (Chapter 3); How are seats *apportioned* in a legislative body? (Chapter 4).

- **PART 2 Management Science.** This part deals with questions of efficiency—how to manage some valuable resource (time, money, energy) so that utility is maximized. How do we sweep over a network with the least amount of backtracking? (Chapter 5); How do we find the shortest or least expensive route that *visits* a specified set of locations? (Chapter 6); How do we create efficient networks that *connect* people or things? (Chapter 7); How do we schedule a project so that it is completed as early as possible? (Chapter 8).
- **PART 3 Growth.** In this part we discuss, in very broad terms, the mathematics of growth and decay, profit and loss. In Chapter 9 we cover mathematical models of *population growth*, mostly biological and human populations but also populations of inanimate "things" such as garbage and pollution. Since money plays such an important role in our lives, it deserves a chapter of its own. In Chapter 10 we discuss a few of the key concepts of *financial mathematics*: interest, investments, retirement savings, and consumer debt.
- **PART 4 Shape and Form.** In this part we cover a few connections between mathematics and the shape and form of objects—natural or human-made. What is *symmetry*? What *types* of symmetries exist in nature and art? (Chapter 11); What kind of geometry lies hidden behind the *kinkiness* of the many irregular shapes we find in nature? (Chapter 12); What is the connection between the *Fibonacci numbers* and the *golden ratio* (two abstract mathematical constructs) and the *spiral* forms that we regularly find in nature? (Chapter 13).
- **PART 5 Statistics.** In one way or another, statistics affects all our lives. Government policy, insurance rates, our health, our diet, and our political lives are all governed by statistical information. This part deals with how the statistical information that affects our lives is collected, organized, and interpreted. What are the purposes and strategies of *data collection*? (Chapter 14); How is data *organized*, *presented,* and *summarized*? (Chapter 15); How do we use mathematics to measure *uncertainty* and *risk*? (Chapter 16); How do we use mathematics to model, analyze, and make predictions about *real-life, bell-shaped* data sets? (Chapter 17).

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A large number of colleagues have contributed both formally and informally to the evolution of this text. (My apologies to anyone whose name has inadvertently been left out.)

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Last, but not least, the *real movers and shakers* in the editorial staff that made this edition possible and deserve special recognition: mover and shaker in-chief (and Senior Acquisitions Editor) Marnie Greenhut, the voice of reason and calm whenever the project hit rough waters; Content Producer Patty Bergin and Project Manager at Cenveo Publisher Services Marilyn Dwyer, both of whose patience, good humor, and attention to detail made the logistics of producing this edition the smoothest ever; Product Marketing Manager Alicia Frankel; Field Marketing Manager Andrew Noble; Designer Barbara Atkinson; and Media Producer Nick Sweeny.

A Final Word to the Reader

"It's not what you look at that matters, it's what you see."
– *Henry David Thoreau*

My goal in writing this text is to shine a small light on all that mathematics can be when looked at in the right way—useful, interesting, subtle, beautiful, and accessible. I hope that you will see something of that in this text.

Peter Tannenbaum

New in This Edition

- New and updated examples from pop culture, sports, politics, and science.
- New material on **Retirement Savings** in Chapter 10.
- New **Applet Bytes** exercises in the exercise sets require the use of the new applets in MyMathLab and encourage students to delve deeper into the concepts using the applets.
- New and updated exercises have been informed by MyMathLab data analytics including level of difficulty and appropriateness.
- New **Annotated Instructor's Edition** provides annotations indicating where Applets, Animated Whiteboard Concept Videos, and Learning Catalytics are relevant, in addition to Discussion Ideas and Teaching Tips. Answers to exercises are still in the back of the book.
- New in MyMathLab for *Excursions in Modern Mathematics*, Ninth Edition
    - New and improved **applets** designed by the author help students visualize the more difficult concepts and develop deeper understanding:
        - Voting Methods
        - Banzhaf and Shapley-Shubik Power
        - Method of Sealed Bids
        - Method of Markers
        - Apportionment Methods
        - Euler Paths and Circuits: Fleury's Algorithm
        - Hamilton Paths and Circuits
        - Traveling Salesman
        - Kruskal's Algorithm
        - Priority List Scheduling
        - Finance Calculator
        - Rigid Motions
        - Geometric Fractals
        - Numerical Summaries of Data

      All applets are assignable in MyMathLab and exercises have been written to guide students.
    - Engaging **Animated Whiteboard Concept Videos** bring concepts to life in an exciting and interesting fashion using narration and animated drawing. Videos cover topics such as Fair Division, Eulerizing Graphs, Self-Similarity, The Golden Ratio, and Normal Curves. Students will see math in a fresh, new way!
    - **Learning Catalytics**, a "bring your own device" student engagement, assessment, and classroom intelligence system, is available in MyMathLab with annotations at point-of-use for instructors in the Annotated Instructor's Edition. LC annotations provide a corresponding code for each question as it becomes relevant to integrate into the classroom. Within Learning Catalytics, simply search for the question using the code in the text's annotation.
    - **StatCrunch,** a powerful, Web-based statistical software that allows users to perform complex analyses, share data sets, and generate compelling reports, has been integrated into the MyMathLab for the first time. The vibrant online community offers tens of thousands of data sets shared by users.

Resources for Success

MyMathLab® Online Course for *Excursions in Modern Mathematics,* 9th Edition by Peter Tannenbaum (access code required)

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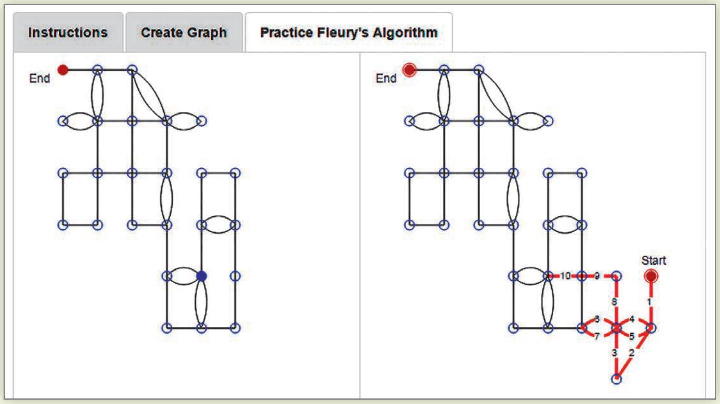
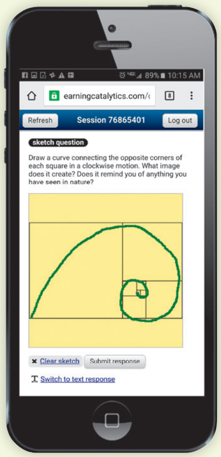


Animated Concept Videos

New Animated Whiteboard Concept videos use narration and animated drawing to bring concepts to life in a more engaging manner for students. Videos cover topics such as Fair Division, Eulerizing Graphs, Self-Similarity, The Golden Ratio, and Normal Curves.



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Resources for Success

Instructor Resources

NEW! Annotated Instructor's Edition

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The AIE provides annotations for instructors, including suggestions about where media resources like Applets and Animated Whiteboard Videos apply, as well as Learning Catalytics questions, discussion ideas, and teaching tips.

The following resources can be downloaded from www.pearsonhighered.com or in MyMathLab.

Instructor's Solutions Manual

Dale R. Buske, St. Cloud State University

This manual contains detailed, worked-out solutions to all exercises in the text.

Instructor's Testing Manual

This manual includes two alternative multiple-choice tests per chapter.

Image Resources Library

This resource in MyMathLab contains all art from the text for instructors to use in their own presentations and handouts.

PowerPoints

These editable slides present key concepts and definitions from the text. You can add art from the Image Resource Library in MyMathLab® or slides that you develop on your own.

TestGen

TestGen® (www.pearsoned.com/testgen) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text.

Student Resources

Student's Solutions Manual

Dale R. Buske, St. Cloud State University

ISBN 10: 0-13-446913-5 **ISBN 13:** 978-0-13-446913-3

This manual provides detailed worked-out solutions to odd-numbered walking and jogging exercises.

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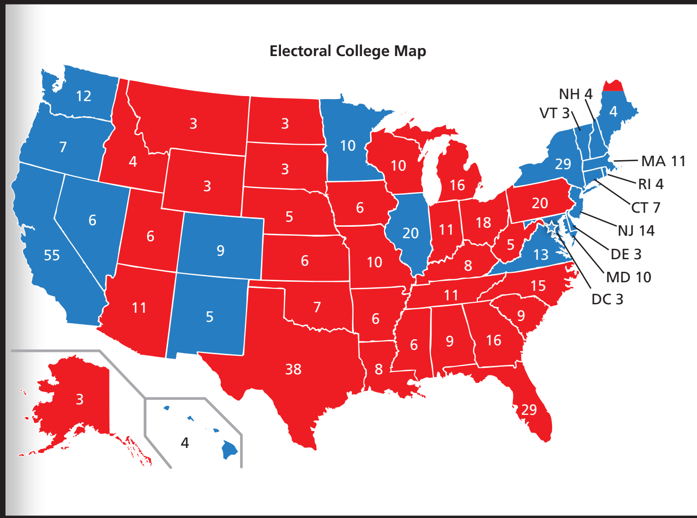
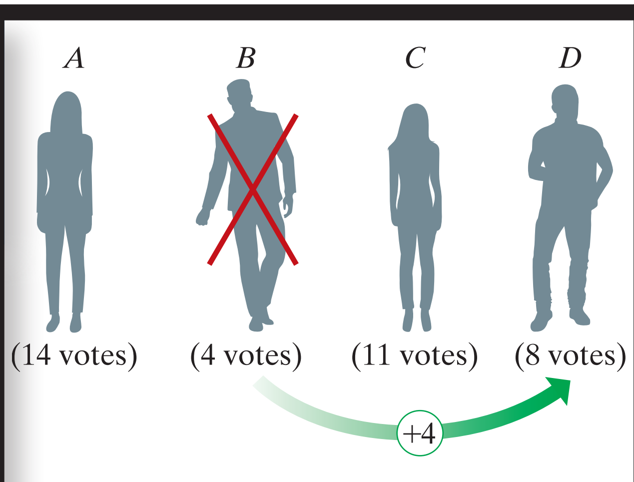
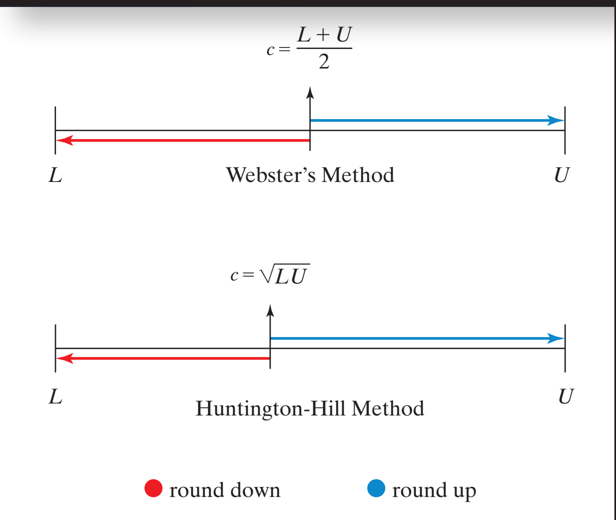
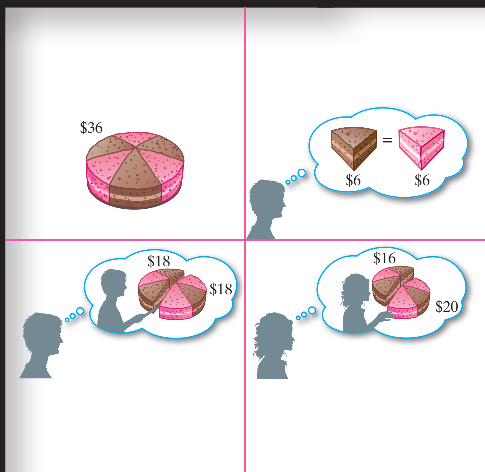
PART 1



Social Choice







2015 Heisman Trophy finalists: Derrick Henry of the University of Alabama, Christian McCaffrey of Stanford University, and Deshaun Watson of Clemson University. (For the full story, see page 13.)

1

The Mathematics of Elections

The Paradoxes of Democracy

Whether we like it or not, we are all affected by the outcomes of elections. Our president, senators, governors, and mayors make decisions that impact our lives in significant ways, and they all get to be in that position because an election made it possible. But elections touch our lives not just in politics. The Academy Awards, Heisman trophies, NCAA football rankings, *American Idol*—they are all decided by some sort of election. Even something as simple as deciding where to go for dinner might require a little family election.

We have elections because we don't all think alike. Since we cannot all have things our way, we vote. But *voting* is only the first half of the story, the one we are most familiar with. As playwright Tom Stoppard suggests, it's the second half of the story—the *counting*—that is at the heart of the democratic process. How do we sift through the many choices of individual voters to find the collective choice of the group? More important, how well does the process work? Is the process always fair? Answering these questions and explaining a few of the many intricacies and subtleties of *voting theory* are the purpose of this chapter.

But wait just a second! Voting theory? Why do we need a fancy theory to figure out how to count the votes? It all sounds pretty simple: We have an election; we count the ballots. Based on that count, we decide the outcome of the election in a consistent and fair manner. Surely, there must be a reasonable way to accomplish this. Surprisingly, there isn't!

In the late 1940's the American economist Kenneth Arrow discovered a remarkable fact: For elections involving three or more candidates, there is no consistently fair democratic method for choosing a winner. In fact, Arrow demonstrated that *a method for determining election results that is always fair is a mathematical impossibility*. This fact, the most famous in voting theory, is known as *Arrow's Impossibility Theorem*.

> "It's not the voting that's democracy; it's the counting."
>
> *– Tom Stoppard*

This chapter is organized as follows. We will start with a general discussion of *elections* and *ballots* in Section 1.1. This discussion provides the backdrop for the remaining sections, which are the heart of the chapter. In Sections 1.2 through 1.5 we will explore four of the most commonly used *voting methods*—how they work and how they are used in real-life applications. In Section 1.6 we will introduce some basic principles of fairness for voting methods and apply these *fairness criteria* to the voting methods discussed in Sections 1.2 through 1.5. The section concludes with a discussion of the meaning and significance of Arrow's Impossibility Theorem.

1.1 The Basic Elements of an Election

Big or small, important or trivial, *all* elections share a common set of elements.

- **The candidates.** The purpose of an election is to choose from a set of *candidates* or *alternatives* (at least two—otherwise it is not a real election). Typically, the word *candidate* is used for people and the word *alternative* is used for other things (movies, football teams, pizza toppings, etc.), but it is acceptable to use the two terms interchangeably. In the case of a generic choice (when we don't know if we are referring to a person or a thing), we will use the term *candidate*. While in theory there is no upper limit on the number of candidates, for most elections (in particular the ones we will discuss in this chapter) the number of candidates is small.
- **The voters.** These are the people who get a say in the outcome of the election. In most democratic elections the presumption is that all voters have an equal say, and we will assume this to be the case in this chapter. (This is not always true, as we will see in great detail in Chapter 2.) The number of voters in an election can range from very small (as few as 3 or 4) to very large (hundreds of millions). In this section we will see examples of both.
- **The ballots.** A ballot is the device by means of which a voter gets to express his or her opinion of the candidates. The most common type is a paper ballot, but a

voice vote, a text message, or a phone call can also serve as a "ballot" (see Example 1.5 *American Idol*). There are many different forms of ballots that can be used in an election, and Fig. 1-1 shows a few common examples. The simplest form is the **single-choice ballot**, shown in Fig. 1-1(a). Here very little is being asked of the voter ("pick the candidate you like best, and keep the rest of your opinions to yourself!"). At the other end of the spectrum is the **preference ballot**, where the voter is asked to rank *all* the candidates in order of preference. Figure 1-1(b) shows a typical preference ballot in an election with five candidates. In this ballot, the voter has entered the candidates' names in order of preference. An alternative version of the same preference ballot is shown in Fig. 1-1(c). Here the names of the candidates are already printed on the ballot and the voter simply has to mark first, second, third, etc. In elections where there are a large number of candidates, a **truncated preference ballot** is often used. In a truncated preference ballot the voter is asked to rank some, but not all, of the candidates. Figure 1-1(d) shows a truncated preference ballot for an election with dozens of candidates.

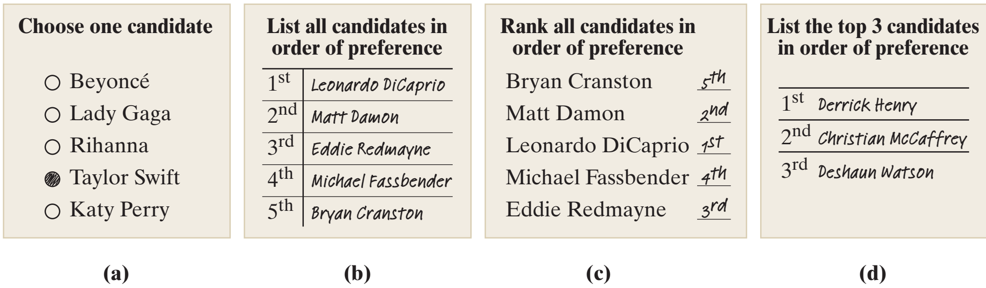


FIGURE 1-1 (a) Single-choice ballot, (b) preference ballot, (c) a different version of the same preference ballot, and (d) truncated preference ballot.

- **The outcome.** The purpose of an election is to use the information provided by the ballots to produce some type of outcome. But what types of outcomes are possible? The most common is **winner-only**. As the name indicates, in a winner-only election all we want is to find a winner. We don't distinguish among the nonwinners. There are, however, situations where we want a broader outcome than just a winner—say we want to determine a first-place, second-place, and third-place candidate from a set of many candidates (but we don't care about fourth place, fifth place, etc.). We call this type of outcome a **partial ranking**. Finally, there are some situations where we want to rank *all* the candidates in order: first, second, third, . . . , last. We call this type of outcome a **full ranking**, or just a **ranking** for short.
- **The voting method.** The final piece of the puzzle is the method that we use to tabulate the ballots and produce the outcome. This is the most interesting (and complicated) part of the story, but we will not dwell on the topic here, as we will discuss voting methods throughout the rest of the chapter.

It is now time to illustrate and clarify the above concepts with some examples. We start with a simple example of a fictitious election. This is an important example, and we will revisit it many times throughout the chapter. You may want to think of Example 1.1 as a mathematical parable, its importance being not in the story itself but in what lies hidden behind it. (As you will soon see, there is a lot more to Example 1.1 than first meets the eye.)

EXAMPLE 1.1 THE MATH CLUB ELECTION (WINNER-ONLY)

The Math Appreciation Society (MAS) is a student club dedicated to an unsung but worthy cause: that of fostering the enjoyment and appreciation of mathematics among college students. The MAS chapter at Tasmania State University is holding its annual election for club president, and there are four *candidates* running: Alisha, Boris, Carmen, and Dave (A, B, C, and D for short).

Every member of the club is eligible to vote, and the vote takes the form of a *preference ballot*. Each voter is asked to rank each of the four candidates in order of preference. There are 37 *voters* who submit their ballots, and the 37 *preference ballots* submitted are shown in Fig. 1-2.

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *A*	1st *B*	1st *A*	1st *C*	1st *B*	1st *C*	1st *A*	1st *B*	1st *C*	1st *A*	1st *C*	1st *D*	1st *A*	1st *A*	1st *C*	1st *A*	1st *C*	1st *D*
2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *B*	2nd *C*	2nd *B*	2nd *B*	2nd *B*	2nd *B*	2nd *B*	2nd *C*
3rd *C*	3rd *C*	3rd *C*	3rd *D*	3rd *C*	3rd *D*	3rd *C*	3rd *C*	3rd *D*	3rd *C*	3rd *D*	3rd *B*	3rd *C*	3rd *C*	3rd *D*	3rd *C*	3rd *D*	3rd *B*
4th *D*	4th *A*	4th *D*	4th *A*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *D*	4th *A*	4th *D*	4th *A*	4th *A*

Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot	Ballot
1st *C*	1st *A*	1st *D*	1st *D*	1st *C*	1st *C*	1st *D*	1st *A*	1st *D*	1st *C*	1st *A*	1st *D*	1st *B*	1st *A*	1st *C*	1st *A*	1st *A*	1st *D*	1st *A*
2nd *B*	2nd *B*	2nd *C*	2nd *C*	2nd *B*	2nd *B*	2nd *C*	2nd *B*	2nd *C*	2nd *B*	2nd *B*	2nd *C*	2nd *D*	2nd *B*	2nd *D*	2nd *B*	2nd *B*	2nd *C*	2nd *B*
3rd *D*	3rd *C*	3rd *B*	3rd *B*	3rd *D*	3rd *D*	3rd *B*	3rd *C*	3rd *B*	3rd *D*	3rd *C*	3rd *B*	3rd *C*	3rd *C*	3rd *B*	3rd *C*	3rd *C*	3rd *B*	3rd *C*
4th *A*	4th *D*	4th *A*	4th *A*	4th *A*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*	4th *A*	4th *D*	4th *A*	4th *D*	4th *D*	4th *A*	4th *D*

FIGURE 1-2 The 37 preference ballots for the Math Club election.

Last but not least, what about the *outcome* of the election? Since the purpose of the election is to choose a club president, it is pointless to discuss or consider which candidate comes in second place, third place, etc. This is a *winner-only* election.

EXAMPLE 1.2 THE MATH CLUB ELECTION (FULL RANKING)

Suppose now that we have pretty much the same situation as in Example 1.1 (same candidates, same voters, same preference ballots), but in this election we have to choose not only a president but also a vice-president, a treasurer, and a secretary. According to the club bylaws, the president is the candidate who comes in first, the vice-president is the candidate who comes in second, the treasurer is the candidate who comes in third, and the secretary is the candidate who comes in fourth. Given that there are four candidates, each candidate will get to be an officer, but there is a big difference between being elected president and being elected treasurer (the president gets status and perks; the treasurer gets to collect the dues and balance the budget). In this version how you place matters, and the outcome should be a full *ranking* of the candidates.

EXAMPLE 1.3 THE ACADEMY AWARDS

The Academy Awards (also known as the Oscars) are given out each year by the Academy of Motion Picture Arts and Sciences for Best Picture, Best Actress, Best Actor, Best Director, and many other, lesser categories (Sound Mixing, Makeup, etc.). The election process is not the same for all awards, so for the sake of simplicity we will just discuss the selection of Best Picture.

The *voters* in this election are all the eligible members of the Academy (a tad over 6000 voting members for the 2016 Academy Awards). After a complicated preliminary round (a process that we won't discuss here) somewhere between eight and ten films are selected as the nominees—these are our *candidates*. (For most other awards there are only five nominees.) Each voter is asked to submit a preference ballot ranking the ten candidates. There is only a winner (the other candidates are not ranked), with the winner determined by a voting method called plurality-with-elimination that we will discuss in detail in Section 1.4. (The winner of the 2016 Best Picture Award was *Spotlight*.)

The part with which people are most familiar comes after the ballots are submitted and tabulated—the annual Academy Awards ceremony, held each year in late February. How many movie fans realize that behind one of the most extravagant and glamorous events in pop culture lies an election?

EXAMPLE 1.4 THE HEISMAN TROPHY

The Heisman Memorial Trophy Award is given annually to the "most outstanding player in collegiate football." The Heisman, as it is usually known, is not only a very prestigious award but also a very controversial award. With so many players playing so many different positions, how do you determine who is the most "outstanding"?

In theory, any football player in any division of college football is a potential *candidate* for the award. In practice, the real candidates are players from Division I programs and are almost always in the glamour positions—quarterback or running back. (Since its inception in 1935, only once has the award gone to a defensive player—Charles Woodson of Michigan.)

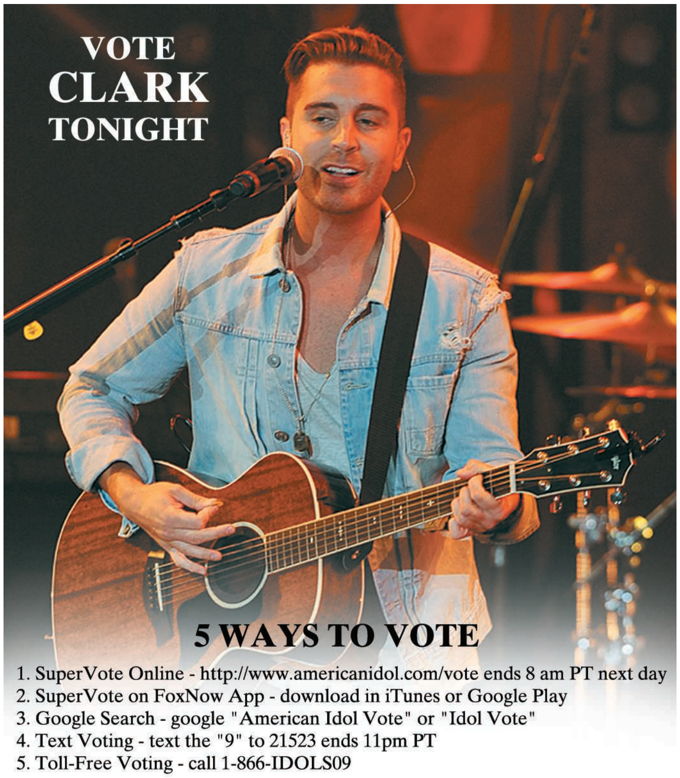
The *voters* are members of the media plus all past Heisman award winners still living, plus one vote from the public (as determined by a survey conducted by ESPN). There are approximately 930 *voters* (the exact number of voters varies each year). Each voter submits a *truncated preference ballot* consisting of a first, second, and third choice (see Fig. 1-1[d]). A first-place vote is worth 3 points, a second-place vote 2 points, and a third-place vote 1 point. The candidate with the most total points from all the ballots is awarded the Heisman trophy in a televised ceremony held each December at the Downtown Athletic Club in New York.

While only one player gets the award, the finalists are ranked by the number of total points received, in effect making the *outcome* of the Heisman trophy a *partial ranking* of the top candidates. (For the 2015 season, the winner was Derrick Henry of the University of Alabama, second place went to Christian McCaffrey of Stanford, and third place went to Deshaun Watson of Clemson.)

EXAMPLE 1.5 *AMERICAN IDOL*

American Idol is a popular reality TV singing competition for individuals. Each year, the winner of *American Idol* gets a big recording contract, and many past winners have gone on to become famous recording artists (Kelly Clarkson, Carrie Underwood, Taylor Hicks). While there is a lot at stake and a big reward for winning, *American Idol* is not a winner-only competition, and there is indeed a ranking of all the finalists. In fact, some nonwinners (Clay Aiken, Jennifer Hudson) have gone on to become great recording artists in their own right.

The 12 (sometimes 13) candidates who reach the final rounds of the competition compete in a weekly televised show. During and immediately after each



weekly show the voters cast their votes. The candidate with the fewest number of votes gets eliminated from the competition, and the following week the process starts all over again with one fewer candidate (occasionally two candidates are eliminated in the same week—see Table 1-11). And who are the *voters* responsible for deciding the fate of these candidates? Anyone and everyone—you, me, Aunt Betsie—we are all potential voters. All one has to do to vote for a particular candidate is to text or call a toll-free number specific to that candidate ("to vote for Clark, call 1-866-IDOLS09," etc.). *American Idol* voting is an example of democracy run amok—you can vote for a candidate even if you never heard her sing, and you can vote as many times as you want.

By the final week of the competition there are only two finalists left, and after one last frenzied round of voting, the winner is determined. (For the final 2016 season of *American Idol* the two finalists were La'Porsha Renae and Trent Harmon, the eventual winner. Table 1-11 shows a detailed summary of the results.)

Examples 1.1 through 1.5 represent just a small sample of how elections can be structured, both in terms of the ballots (think of these as the *inputs* to the election) and the types of outcomes we look for (the *outputs* of the election). We will revisit some of these examples and many others as we wind our way through the chapter.

Preference Ballots and Preference Schedules

Let's focus now on elections where the balloting is done by means of preference ballots, as in Examples 1.1 and 1.2. The great advantage of preference ballots (compared with, for example, single-choice ballots) is that they provide a great deal of useful information about an individual voter's preferences—in both direct and indirect ways.

To illustrate what we mean, consider the preference ballot shown in Fig. 1-3. This ballot directly tells us that the voter likes candidate *C* best, *B* second best, *D* third best, and *A* last. But, in fact, this ballot tells us a lot more—it tells us unequivocally which candidate the voter would choose if it came down to a choice between just two candidates. For example, if it came down to a choice between, say, *A* and *B*, which one would this voter choose? Of course she would choose *B*—she has *B* above *A* in her ranking. Thus, a preference ballot allows us to make relative comparisons between any two candidates—*the candidate higher on the ballot is always preferred over the candidate in the lower position*. Please take note of this simple but important idea, as we will use it repeatedly later in the chapter.

Ballot
1st *C*
2nd *B*
3rd *D*
4th *A*

FIGURE 1-3

The second important idea we will use later is the assumption that the relative preferences in a preference ballot do not change if one of the candidates withdraws or is eliminated. Once again, we can illustrate this using Fig. 1-3. What would happen if for some unforeseen reason candidate *B* drops out of the race right before the ballots are tabulated? Do we have to have a new election? Absolutely not—the old ballot simply becomes the ballot shown on the right side of Fig. 1-4. The candidates above *B* stay put and each of the candidates below *B* moves up a spot.

Ballot		**Ballot**
1st *C*		1st *C*
~~2nd *B*~~	→	2nd *D*
3rd *D*		3rd *A*
4th *A*		

FIGURE 1-4